Dispersive estimates for the three-dimensional Schrödinger equation with rough potentials

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Abstract

The three-dimensional Schrödinger propogator e^{itH}, $H = -\triangle + V$, is a bounded map from L^1 to L^∞ with norm controlled by $|t|^{-3/2}$ provided the potential satisfies two conditions: An integrability condition limiting the singularities and decay of V, and a zero-energy spectral condition on H. This is shown by expressing the spectral measure of H in terms of its resolvents and proving a family of L^p mapping estimates for the resolvents. Previous results in this direction had required V to satisfy explicit pointwise bounds.

1 Introduction

In this paper we consider dispersive estimates for the the time evolution operator $e^{itH}P_{ac}(H)$, where $H = -\Delta + V$ in $\mathbb{R}^3$ and $P_{ac}(H)$ is the projection onto the absolutely continuous subspace of H. Our goal is to assume as little as possible on the potential $V = V(x)$ in terms of decay or regularity. More precisely, we prove the following theorem.

Theorem 1. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Assume also that zero is neither an eigenvalue nor a resonance of $H = -\triangle + V$. Then*

$$\left\| e^{itH} P_{ac}(H) \right\|_{1\to\infty} \lesssim |t|^{-\frac{3}{2}}. \tag{1}$$

See Section 3 for a discussion of resonances. With this assumption the spectrum is known to be purely absolutely continuous on $[0,\infty)$, see [GS2] for details.

Previous results in this direction have generally required pointwise decay of V. Journé, Soffer and Sogge [JSS] proved a version of Theorem 1 under the pointwise bound $|V(x)| \le C(1+|x|)^{-\beta}$, $\beta > 7$, and also some regularity assumptions including $\hat{V} \in L^1$. Yajima [Yaj] reduced the decay hypothesis to $\beta > 5$ and proved that the wave operators are bounded on $L^p(\mathbb{R}^3)$ for all $1 \le p \le \infty$. The dispersive estimate follows from this result. Finally, Goldberg and Schlag [GS1] established the dispersive estimate provided $\beta > 3$. In all these works the assumption is made that zero energy is neither an eigenvalue nor a resonance.

The exposition in this paper roughly follows [GS1], with two significant refinements. First, the distinction which was previously drawn between high and low energies is now removed. Second, the limiting absorption principle of Agmon [Ag], which concerns the action of resolvents on weighted L^2, is replaced with unweighted L^p estimates as in [GS2]. The ability to work with potentials that satisfy L^p conditions (but not necessarily any pointwise bounds) depends in turn on a unique continuation result due to Ionescu and Jerison [IonJer]. For reference we present the statement here.

Theorem 2. *Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Suppose $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies $(-\triangle + V)u = \lambda^2 u$ where $\lambda \neq 0$ in the sense of distributions. If, moreover, $\|(1+|x|)^{\delta - \frac{1}{2}} u\|_2 < \infty$ for some $\delta > 0$, then $u \equiv 0$.*

In terms of local regularity, Theorem 1 appears to be nearly optimal. There exist compactly supported potentials $V \in L^{3/2}_{\mathrm{weak}}$ for which $-\Delta + V$ admits bound states with positive energy [KocTat]. On the other hand, while the assumption $V \in L^1(\mathbb{R}^3)$ corresponds to (radial) pointwise decay on the order of $|V(x)| \leq C(1+|x|)^{-3-\varepsilon}$, it is reasonable to expect dispersive behavior to persist even with weaker decay hypotheses on V. This is already shown in [RodSch] for small potentials in the Kato class, which includes all $V \in L^{\frac{3}{2}+\varepsilon} \cap L^{\frac{3}{2}-\varepsilon}$ of small norm.

1.1 Resolvent Identities

Let $H = -\Delta + V$ in $\mathbb{R}^3$ and define the resolvents $R_0(z) := (-\Delta - z)^{-1}$ and $R_V(z) := (H - z)^{-1}$. For $z \in \mathbb{C} \setminus \mathbb{R}^+$, the operator $R_0(z)$ can be realized as an integral operator with the kernel

$$R_0(z)(x,y) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|}$$

where $\sqrt{z}$ is taken to have positive imaginary part. While $R_V(z)$ does not possess an explicit representation of this form, it can be expressed in terms of $R_0(z)$ via the identities

$$\begin{aligned} R_V(z) &= (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1} \\ R_V(z) &= R_0(z) - R_0(z)VR_V(z) = R_0(z) - R_V(z)VR_0(z) \end{aligned} \tag{2}$$

In the case where $z = \lambda \in \mathbb{R}^+$, one is led to consider limits of the form $R_0(\lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon)$. The choice of sign determines which branch of the square-root function is selected in the formula above, therefore the two continuations do not agree with one another. For convenience we will adopt a shorthand notation for dealing with resolvents along the positive real axis, namely

$$\begin{aligned} R_0^{\pm}(\lambda) &:= R_0(\lambda \pm i0) \\ R_V^{\pm}(\lambda) &:= R_V(\lambda \pm i0) \end{aligned}$$

Note that $R_0^-(\lambda)$ is the formal adjoint of $R_0^+(\lambda)$, and a similar relationship holds for $R_V^{\pm}(\lambda^2)$. The discrepancy between $R_0^+(\lambda)$ and $R_0^-(\lambda)$ characterizes the absolutely continuous part of the spectral measure of H, denoted here by $E_{ac}(d\lambda)$, by means of the Stone formula

$$\langle E_{ac}(d\lambda)f, g\rangle = \frac{1}{2\pi i}\big\langle [R_V^+(\lambda) - R_V^-(\lambda)]f, g\big\rangle \, d\lambda. \tag{3}$$

Let χ be a smooth, even, cut-off function on the line that is equal to one on a neighborhood of the origin. In order to prove Theorem 1 it will suffice to show that

$$\begin{aligned} \sup_{L\geq 1}\Big|\Big\langle e^{itH}\chi(\sqrt{H}/L)P_{a.c.}f, g\Big\rangle\Big| &= \sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda^2}\lambda\,\chi(\lambda/L)\Big\langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f, g\Big\rangle \frac{d\lambda}{\pi i}\Big| \\ &\lesssim |t|^{-\frac{3}{2}}\|f\|_1\|g\|_1. \end{aligned} \tag{4}$$

The first equality is precisely (3), and we have also made the change of variable $\lambda \mapsto \lambda^2$.

Our approach roughly parallels the one found in [GS1], with two main differences. The first is that norms will be estimated in a variety of L^p spaces in addition to the more typical weighted L^2. The second is that low and high energies will not require a separate calculation. There is still a distinction to be noted between the two cases, however. The limiting absorption principle is used to establish decay as $\lambda \to \infty$, whereas boundedness at low energies follows from a Fredholm alternative argument. This requires assuming that zero is neither an eigenvalue nor a resonance.

1.2 Initial terms of the Born series

Iterating the resolvent identity (2) a total of $m+2$ times yields the finite Born series

$$\begin{aligned} R_V^{\pm}(\lambda^2) = &\sum_{k=0}^{m+1} R_0^{\pm}(\lambda^2)(-VR_0^{\pm}(\lambda^2))^k \\ &+ R_0^{\pm}(\lambda^2)VR_V^{\pm}(\lambda^2)(VR_0^{\pm}(\lambda^2))^{m+1}. \end{aligned} \tag{5}$$

Here m is any positive integer. This expansion is then inserted into the integral in (4). The first $m+2$ terms which do not contain the resolvent R_V are treated as in [RodSch], Section 2, which only requires that

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3} \int \frac{|V(y)|}{|x-y|}\,dy < \infty. \tag{6}$$

In particular, if $V \in L^{\frac{3}{2}+\varepsilon} \cap L^{\frac{3}{2}-\varepsilon}$, then this condition is satisfied by dividing $\mathbb{R}^3$ into the regions $|x-y|<1$ and $|x-y|\geq 1$.

For the convenience of the reader we recall the relevant arguments from [RodSch]. When the Born series (5) is substituted into (4), the contricuation from the k^{th} term is equal to

$$\int_0^\infty e^{it\lambda^2}\lambda\,\psi(\lambda/L)\left\langle \left[R_0^+(\lambda^2)(VR_0^+(\lambda^2))^k - R_0^-(\lambda^2)(VR_0^-(\lambda^2))^k\right] f,g\right\rangle d\lambda$$

which is controlled by

$$\begin{aligned} &\sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda^2}\lambda\,\psi(\lambda/L)\,\Im\langle R_0^+(\lambda^2)(VR_0^+(\lambda^2))^k f,g\rangle\,d\lambda\Big| \\ &\leq \int_{\mathbb{R}^6}|f(x_0)||g(x_{k+1})|\int_{\mathbb{R}^{3k}}\frac{\prod_{j=1}^k|V(x_j)|}{\prod_{j=0}^k 4\pi|x_j-x_{j+1}|}\cdot \end{aligned} \tag{7}$$

$$\cdot\sup_{L\geq 1}\Big|\int_0^\infty e^{it\lambda^2}\lambda\,\psi(\lambda/L)\,\sin\Big(\lambda\sum_{\ell=0}^k|x_\ell-x_{\ell+1}|\Big)\,d\lambda\Big|\,d(x_1,\ldots,x_k)\,dx_0\,dx_{k+1} \tag{8}$$

$$\leq Ct^{-\frac{3}{2}}\int_{\mathbb{R}^6}|f(x_0)||g(x_{k+1})|\int_{\mathbb{R}^{3k}}\frac{\prod_{j=1}^k|V(x_j)|}{(4\pi)^{k+1}\prod_{j=0}^k|x_j-x_{j+1}|}\sum_{\ell=0}^k|x_\ell-x_{\ell+1}|\,d(x_1,\ldots,x_k)\,dx_0\,dx_{k+1} \tag{9}$$

which in turn is controlled by

$$\begin{aligned}(10) \qquad & \leq C t^{-\frac{3}{2}} \int_{\mathbb{R}^6} |f(x_0)||g(x_{k+1})| \, (k+1)(\|V\|_{\mathcal{K}}/4\pi)^k \, dx_0 \, dx_{k+1} \\ & \leq C_k \, t^{-\frac{3}{2}} \|f\|_1 \|g\|_1 .\end{aligned}$$

In order to pass to (7) one uses the explicit representation of the kernel of $R_0^+(\lambda^2)(x,y) = \frac{e^{i\lambda|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$, which leads to a k-fold integral. The inequalities (8) and (10) are obtained by means of the following two lemmas from [RodSch], which we reproduce here without proof. They may be regarded as exercises in the use of stationary phase and Fubini's Theorem, respectively.

Lemma 3. *Let ψ be a smooth, even bump function with $\psi(\lambda) = 1$ for $-1 \leq \lambda \leq 1$ and $\mathrm{supp}(\psi) \subset [-2,2]$. Then for all $t \geq 1$ and any real a,*

$$(11) \qquad \sup_{L \geq 1} \Big| \int_0^\infty e^{it\lambda} \sin(a\sqrt{\lambda}) \, \psi\Big(\frac{\sqrt{\lambda}}{L}\Big) \, d\lambda \Big| \leq C \, t^{-\frac{3}{2}} \, |a|$$

where C only depends on ψ and χ.

Lemma 4. *For any positive integer k and V as in* (6)

$$\sup_{x_0, x_{k+1} \in \mathbb{R}^3} \int_{\mathbb{R}^{3k}} \frac{\prod_{j=1}^k |V(x_j)|}{\prod_{j=0}^k |x_j - x_{j+1}|} \sum_{\ell=0}^k |x_\ell - x_{\ell+1}| \, dx_1 \dots dx_k \leq (k+1) \|V\|_{\mathcal{K}}^k .$$

2 Estimates on the free resolvent

We now turn to the term in the Born series (5) containing the perturbed resolvent R_V. The following propositions establish a family of mapping estimates for the free resolvent.

Proposition 5. *For each exponent $1 < p \leq \frac{4}{3}$, there exist constants $C_p < \infty$ such that*

$$\|R_0^{\pm}(\lambda^2) f\|_{L^{3p}} \leq C_p \lambda^{-2+2/p} \|f\|_{L^p}$$

For each exponent $\frac{4}{3} \leq p < \frac{3}{2}$, there exist constants $C_p < \infty$ such that

$$\|R_0^{\mp}(\lambda^2) f\|_{L^{p*}} \leq C_p \lambda^{4-6/p} \|f\|_{L^p} \qquad \text{where } \frac{1}{p*} = \frac{3}{p} - 2$$

Proof. The case $p = \frac{4}{3}$ is proven as a special case of theorem 2.3 in [KRS]. It is clear from fractional integration that $R_0^{\pm}(\lambda^2)$ maps $L^1(\mathbb{R}^3)$ to weak-$L^3(\mathbb{R}^3)$ uniformly in λ, using the definition

$$\|f\|_{L^3_{\mathrm{weak}}(\mathbb{R}^3)} = \sup_{A \subset \mathbb{R}^3, |A| < \infty} |A|^{-2/3} \int_A |f(x)| \, dx$$

which is equivalent to the usual weak-L^3 "norm" and also satisfies a triangle inequality, see Lieb, Loss [LieLos], Chapter 4.3. The cases $1 < p < \frac{4}{3}$ and $\frac{4}{3} < p < \frac{3}{2}$ follow by Marcinkiewicz interpolation and duality, respectively. ∎

Proposition 6. *Suppose $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^{\frac{3}{2}(1-\varepsilon)}(\mathbb{R}^3)$. Then there exists a constant $C_\varepsilon < \infty$ such that*

$$\|VR_0^{\pm}(\lambda^2)f\|_{L^p} \le C_\epsilon(1+|\lambda|)^{-2+2/p}\|V\|\|f\|_{L^p} \quad \textit{for all exponents} \quad 1 \le p \le 1+\varepsilon. \tag{12}$$

The dual operators satisfy the related bound

$$\|R_0^{\pm}(\lambda^2)Vf\|_{L^p} \le C_\varepsilon(1+|\lambda|)^{-2/p}\|V\|\|f\|_{L^p} \quad \textit{for all exponents} \quad \frac{1+\varepsilon}{\varepsilon} \le p \le \infty. \tag{12'}$$

In the above statement $\|V\|$ is understood to be the larger of $\|V\|_{L^{\frac{3}{2}(1+\varepsilon)}}$ and $\|V\|_{L^{\frac{3}{2}(1-\varepsilon)}}$.

Proof. In the case $p = 1$, $(VR_0^{\pm}(\lambda^2))$ has an operator bound of precisely $(4\pi)^{-1}\|V\|_{\mathcal{K}}$, which is controlled by $\|V\|$. The case $p = 1+\epsilon$, $|\lambda| > 1$, is a corollary of the preceding proposition, using the fact that $V \in L^{\frac{3p}{2}}$. For $|\lambda| \le 1$, a uniform bound is obtained by comparing $R_0(\lambda^2)$ to fractional integration and observing that $V \in L^{\frac{3}{2}}$. The intermediate cases $1 < p < 1+\varepsilon$ follow by interpolation. $\square$

It is slightly inconvenient that $\|VR_0^{\pm}(\lambda^2)\|_{1\to 1}$ does not decay in the limit $|\lambda| \to \infty$. If this map is iterated several times, however, we may use the fact that $(VR_0^{\pm}(\lambda^2))$ maps $L^1(\mathbb{R}^3)$ to $L^{2/(2-\varepsilon)}(\mathbb{R}^3)$ and vice versa to apply the bound in (12) with $p = \frac{2}{2-\varepsilon}$. The resulting mapping estimates will be needed in section 4.

Corollary 7. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^{\frac{3}{2}(1-\varepsilon)}(\mathbb{R}^3)$. Then*

$$\begin{aligned} \|(VR_0^{\pm}(\lambda^2))^{k+2}f\|_{L^1} &\le C^k(1+|\lambda|)^{k\varepsilon}\|f\|_{L^1} \qquad \textit{and} \\ \|(R_0^{\pm}(\lambda^2)V)^{k+2}f\|_{L^\infty} &\le C^k(1+|\lambda|)^{k\varepsilon}\|f\|_{L^\infty} \end{aligned} \tag{13}$$

We now consider the action of the free resolvent on weighted L^p spaces. Let $L^{p,\sigma}(\mathbb{R}^3)$ be the Banach space determined by the norm

$$\|f\|_{L^{p,\sigma}} = \|(1+|\cdot|)^\sigma f\|_{L^p}, \quad 1 \le p \le \infty, \sigma \in \mathbb{R}$$

Lemma 8. *Suppose $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and let p be any exponent in the range $\frac{1+\varepsilon}{\varepsilon} \le p \le \infty$. The operator $R_0^{\pm}(\lambda^2)V$ is a bounded linear map on $L^{p,-1}$, and its norm is controlled by $\|V\|$.*

Furthermore, $R_0^{\pm}(\lambda^2)V$ may be written as a sum of linear maps T_1 and T_2 satisfying the following estimates:

$$\|T_1 f\|_{L^{p,-1}} \lesssim (1+|\lambda|)^{-1/p}\|V\|\|f\|_{L^{p,-1}} \tag{14}$$

$$\|T_2 f\|_{L^\infty} \lesssim \|V\|\|f\|_{L^{p,-1}(\{|x|>\lambda^{1/p}\})} \tag{15}$$

The constant of similarity depends on $\varepsilon > 0$ but not on the specific choice of p.

Proof. For each $k = 0, 1, 2, \ldots$, let $D_k = \{x \in \mathbb{R}^3 : |x| < \lambda^{1/p}2^{k+1}\}$. We define T_1 and T_2 in the following manner: In the annulus $A_k = \{x : 2^{k-1} \le |x| < 2^k\}$ (or the unit ball $A_0 = \{|x| < 1\}$), let

$$\begin{aligned} T_1 f(x) &= R_0^{\pm}(\lambda^2)V\chi_{D_k} f(x) \\ T_2 f(x) &= R_0^{\pm}(\lambda^2)V\chi_{D_k^c} f(x) \end{aligned}$$

The estimate for $T_2 f$ is immediate. Since $V \in L^{p'}$, by Hölder's inequality $Vf \in L^{1,-1}$. The construction of D_k ensures that $|y-x| > (1+|y|)/3$ for any $x \in A_k$, $y \in D_k^c$. Thus

$$|T_2 f(x)| \;<\; \frac{3}{4\pi}\int_{D_k^c} \frac{|V(y)f(y)|}{1+|y|}\,dy \;<\; \frac{3}{4\pi}\|Vf\|_{L^{1,-1}(D_0^c)} \;\lesssim\; \|V\|\|f\|_{L^{p,-1}(D_0^c)}$$

It should be noted that $L^\infty(\mathbb{R}^3)$ has a natural embedding into $L^{p,-1}(\mathbb{R}^3)$ for any $p > 3$.

To control $T_1 f$, we first consider its restriction to each annulus A_k. Proposition 6 states that $\|T_1 f\|_{L^p(A_k)} \lesssim \|R_0^\pm(\lambda^2)V\chi_{D_k} f\|_{L^p} \lesssim (1+|\lambda|)^{-2/p}\|V\|\|f\|_{L^p(D_k)}$. The norm of $T_1 f$, as measured in the space $L^{p,-1}(\mathbb{R}^3)$,is recovered by summing over k.

$$\begin{aligned}\|T_1 f\|_{L^{p,-1}}^p \;&\sim\; \sum_{k=0}^\infty 2^{-kp}\|T_1 f\|_{L^p(A_k)}^p \\ &\lesssim\; (1+|\lambda|)^{-2}\|V\|^p \sum_{k=0}^\infty 2^{-kp}\int_{D_k} |f(x)|^p\,dx\end{aligned}$$

Interchange the summation and integral by Fubini's theorem. At each point $x \in \mathbb{R}^3$, $x \in D_k$ only if $k > \log(|x|/(2\lambda^{1/p}))$, so only these terms of the sum will be nonzero. The resulting sum over k is a geometric series with ratio less than $\frac{1}{2}$, which can be estimated by the largest term. Thus

$$\begin{aligned}\|T_1 f\|_{L^{p,-1}}^p \;&\lesssim\; (1+|\lambda|)^{-2}\|V\|^p \int_{\mathbb{R}^3} |f(x)|^p \min(2^p\lambda|x|^{-p}, 1)\,dx \\ &\lesssim 2^p(1+|\lambda|)^{-1}\|V\|^p \int_{\mathbb{R}^3} |f(x)|^p (1+|x|)^{-p}\,dx\end{aligned}$$

Taking p^{th} roots yields the desired conclusion. $\square$

Corollary 9. *Suppose $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and assume $1 \le p \le 1+\varepsilon$. Then $VR_0^\pm(\lambda^2)$ is a bounded operator on $L^{p,1}(\mathbb{R}^3)$ whose norm is controlled by ε and $\|V\|$ alone.*

Proof. This is the dual statement of Lemma 8, since V is real-valued and $L^{p',-\sigma}(\mathbb{R}^3)$ is the space dual to $L^{p,\sigma}(\mathbb{R}^3)$. $\square$

Corollary 10. *Suppose $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and let $p \ge \frac{1+\varepsilon}{\varepsilon}$. Then for all $\lambda \in \mathbb{R}$,*

$$\|(R_0^\pm(\lambda^2)V)^2 f\|_{L^{p,-1}} \lesssim (1+|\lambda|)^{-1/p}\|V\|^2\|f\|_{L^{p,-1}} \tag{16}$$

The dual operators satisfy the bound

$$\|(VR_0^\pm(\lambda^2))^2 f\|_{L^{p,1}} \lesssim (1+|\lambda|)^{-1/p'}\|V\|^2\|f\|_{L^{p,1}} \quad \textit{for all exponents } 1 \le p \le 1+\varepsilon \tag{16'}$$

Proof. This is an estimate on $(T_1+T_2)(T_1+T_2)f$. Any product which includes T_1 will have the desired decay (or better) by Lemma 8. On the other hand,

$$\begin{aligned}\|T_2T_2 f\|_{L^\infty} \;\lesssim\; \|V\|\|T_2 f\|_{L^{\infty,-1}(\mathbb{R}^3\setminus B(0,\lambda^{1/p}))} \;&\lesssim\; (1+|\lambda|)^{-1/p}\|V\|\|T_2 f\|_{L^\infty} \\ &\lesssim\; (1+|\lambda|)^{-1/p}\|V\|^2\|f\|_{L^{p,-1}}\end{aligned}$$

$\square$

This is a crucial estimate for two reasons. First, it guarantees convergence of the Neumann series for $(I + R_0^\pm(\lambda^2)V)^{-1}$ for sufficiently large λ, along with the uniform size bound

$$\limsup_{\lambda\to\infty} \|(I + R_0^\pm(\lambda^2)V)^{-1}\| \lesssim 1 + \limsup_{\lambda\to\infty} \|R_0^\pm(\lambda^2)V\|$$

as measured in the operator norm on $L^{p,-1}(\mathbb{R}^3)$. Second, we will eventually perform an integration by parts in the λ variable, whose boundary terms will vanish because of (16).

3 Estimates on the perturbed resolvent

Recall that the perturbed resolvent $R_V(z)$ is related to the $R_0(z)$ by the identity

$$R_V(z) = (I + R_0(z)V)^{-1}R_0(z) \tag{2}$$

In order to prove that $R_V^\pm(\lambda^2)$ satisfies the same mapping estimates as $R_0^\pm(\lambda^2)$, it therefore suffices to show that $(I + R_0^\pm(\lambda^2)V)^{-1}$ is a bounded operator on the appropriate space. As mentioned above, for large λ this can be done easily by expressing the inverse as a (convergent) power series.

If λ is not large, invertability of $I + R_0^\pm(\lambda^2)V$ is established by a Fredholm-alternative argument. One needs to verify two things: that $I + R_0^\pm(\lambda^2)V$ is a compact perturbation of the identity, and that its null space contains no nonzero elements. This step will require the assumption that zero energy is neither an eigenvalue nor a resonance, so we must first state a precise definition.

Definition 11. *We say that a resonance occurs at zero if the equation $(I + R_0(0)V)f = 0$ admits a distributional solution f such that $f \in L^{2,\sigma}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ for every $\sigma < -\frac{1}{2}$.*

Lemma 12. *Suppose $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and let $\frac{1+\varepsilon}{\varepsilon} \le p \le \infty$. For any fixed $\lambda \in \mathbb{R}$, $R_0^\pm(\lambda^2)V$ is a compact operator mapping $L^{p,-1}(\mathbb{R}^3)$ to itself. By duality, $VR_0^\mp(\lambda^2)$ is a compact operator on $L^{p',1}(\mathbb{R}^3)$.*

Proof. First consider the case where V is bounded with maximum size M and supported in the ball $B(0,R)$. On the support of V, f is integrable with bound $\|f\|_{L^1(\mathrm{supp}(V))} \lesssim R^{1+3/p'}\|f\|_{L^{p,-1}}$. Then for all $|x| > 2R$,

$$|R_0^\pm(\lambda^2)Vf(x)| \lesssim \big(|Vf| * \frac{1}{|\cdot|}\big)(x) \lesssim MR^{1+3/p'}\|f\|_{L^{p,-1}}\,|x|^{-1}$$

Let ψ be a smooth bump function with support in $B(0,2)$ so that $\psi(x) = 1$ whenever $|x| \le 1$, and define $\psi_{\tilde R}(x) = \psi(x/\tilde R)$. If $\tilde R > 2R$, a simple integration yields

$$\lim_{\tilde R\to\infty} \|(1-\psi_{\tilde R})R_0^\pm(\lambda^2)Vf\|_{L^{p,-1}} \lesssim \lim_{\tilde R\to\infty} \big(MR^{1+3/p'}\|f\|_{L^{p,-1}}\big)\tilde R^{1-3/p'} = 0$$

The resolvent tends to increase regularity; for Schwartz functions f we have

$$\begin{aligned}(-\Delta+1)R_0^\pm(\lambda^2)Vf &= (-\Delta-\lambda^2)R_0^\pm(\lambda^2)Vf + (1+\lambda^2)R_0^\pm(\lambda^2)Vf \\ &= Vf + (1+\lambda^2)R_0^\pm(\lambda^2)Vf\end{aligned}$$

which implies that $\|(-\Delta+1)R_0^\pm(\lambda^2)Vf\|_{L^{p,-1}} \lesssim \|f\|_{L^{p,-1}}$. Boundedness of V is also used in this step. Taking limits, the inequality can be extended to all $f \in L^{p,-1}$.

On the compact set $\{|x| \le 2\tilde{R}\}$ the norms L^p and $L^{p,-1}$ are equivalent. Therefore $\psi_{\tilde{R}} R_0^\pm(\lambda^2)V$ is a continuous map from $L^{p,-1}$ to the Sobolev space $W^{2,p}(B(0,2\tilde{R}))$, which embeds compactly into $L^p(B(0,2\tilde{R}))$, and hence also $L^{p,-1}(\mathbb{R}^3)$, by Rellich's theorem.

We have shown that $R_0^\pm(\lambda^2)V$ is a norm-limit of the operators $\psi_{\tilde{R}} R_0^\pm(\lambda^2)V$ as $\tilde{R} \to \infty$, and that each element of this approximating sequence is compact. The set of compact linear operator is closed in the norm topology, so $R_0^\pm(\lambda^2)V$ must be compact as well.

For general potentials $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, it is possible write V as a norm-limit of bounded functions V_n with compact support. For each $n = 1, 2, \ldots$, $R_0^\pm(\lambda^2)V_n$ is a compact operator. The lemma is now proved by another limiting argument, this time with the help of lemma 8. $\square$

Lemma 13. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and $1 \le p \le 1+\varepsilon$. Assume that zero is neither an eigenvalue nor a resonance of $(-\Delta + V)$. Then $(I + VR_0^\pm(\lambda^2))^{-1}$ exists as a bounded linear map on $L^{p,1}(\mathbb{R}^3)$ for all $\lambda \in \mathbb{R}$. By duality, $(I + R_0^\pm(\lambda^2)V)^{-1}$ exists as a bounded operator on $L^{p,-1}(\mathbb{R}^3)$.*

Proof. By lemma 12 and the Fredholm alternative, $I + VR_0^\pm(\lambda^2)$ will fail be invertible only if there exists a function $g \in L^{p,1}(\mathbb{R}^3)$ satisfying $g = -VR_0^\pm(\lambda^2)g$. In fact any such solution g must possess greater regularity than the assumed $g \in L^p_{\text{loc}}$. This is seen by iterating the map $VR_0^\pm(\lambda^2)$.

First note that any function in $L^{q,1}(\mathbb{R}^3), q < \frac{3}{2}$, is integrable. Decompose $R_0^\pm(\lambda^2) = S_1 + S_2$ in the following manner: For $x \in A_k,\ k = 1, 2, \ldots$, let

$$S_1 f(x) = R_0^\pm(\lambda^2)\chi_{\{|x|>2^{k-2}\}} f(x)$$
$$S_2 f(x) = R_0^\pm(\lambda^2)\chi_{\{|x|\le 2^{k-2}\}} f(x)$$

and $S_1 g(x) = R_0^\pm(\lambda^2)g(x)$ if $x \in A_0$. Here, as in lemma 8, A_k denotes an annulus where $|x| \sim 2^k$. One immediately obtains a pointwise estimate for S_2, namely $|S_2 f(x)| \lesssim \|f\|_{L^1}(1+|x|)^{-1}$.

We will see that S_1 is a bounded map from $L^{q,1}(\mathbb{R}^3)$ to $L^{r,1}(\mathbb{R}^3)$, where $\frac{1}{r} = \frac{1}{q} - \frac{2}{3}$ is the exponent given by fractional integration. The calculation is similar to the one in lemma 8, with one additional step to deal with the fact that $r \ne q$.

$$\begin{aligned} \|S_1 g\|_{L^{r,1}} \sim \Big(\sum_{k=0}^\infty 2^{kr}\|S_1 g\|^r_{L^r(A_k)}\Big)^{1/r} &\lesssim \Big(\sum_{k=0}^\infty 2^{kr}\Big(\int_{|x|\ge 2^{k-2}} |g(x)|^q\,dx\Big)^{r/q}\Big)^{1/r} \\ &\le \Big(\int_{\mathbb{R}^3} |g(x)|^q \Big(\sum_{k \le \log 4|x|} 2^{kr}\Big)^{q/r} dx\Big)^{1/q} \\ &\lesssim \Big(\int_{\mathbb{R}^3} |g(x)|^q |x|^q\,dx\Big)^{1/q} = \|g\|_{L^{q,1}} \end{aligned} \tag{17}$$

The exchange of summation and integration is done via Minkowski's inequality, noting that $r > q$.

Putting the two pieces S_1 and S_2 together, we conclude that $R_0^\pm(\lambda^2)$ is a bounded map from $L^{q,1}(\mathbb{R}^3)$ to $L^{r,1}(\mathbb{R}^3) + L^{\infty,1}(\mathbb{R}^3)$. Therefore, if $g \in L^{q,1}(\mathbb{R}^3)$, one can bootstrap in two directions:

$$VR_0^\pm(\lambda^2)g \in L^{1,1}(\mathbb{R}^3) \qquad \text{and} \qquad VR_0^\pm(\lambda^2)g \in L^{\tilde{q},1}(\mathbb{R}^3), \quad \text{where } \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{2\varepsilon}{3(1+\varepsilon)} \tag{18}$$

by estimating $\|V\|$ in $L^{r'} \cap L^1$ and $L^{\frac{3}{2}(1+\varepsilon)} \cap L^{\tilde{q}}$, respectively.

This process terminates once it is established that $g \in L^{1,1}(\mathbb{R}^3) \cap L^{\frac{3}{2}^+,1}(\mathbb{R}^3)$. Consequently, $g \in L^{\frac{3}{2}^+}(\mathbb{R}^3) \cap L^{\frac{3}{2}^-}(\mathbb{R}^3)$, and $R_0^{\mp}(\lambda^2)g \in L^\infty(\mathbb{R}^3)$, which embeds naturally in $L^{\infty,-1}(\mathbb{R}^3)$. The pairing of functions in dual spaces

$$\langle R_0^{\pm}(\lambda^2)g, g\rangle \;=\; -\langle R_0^{\pm}(\lambda^2)g, VR_0^{\pm}(\lambda^2)g\rangle$$

is then well-defined. Furthermore, since V is asumed to be real-valued, the expression on the right side has no imaginary part. On the other hand, by Parseval's identity

$$\Im\langle \mathbb{R}_0^{\pm}(\lambda^2)g, g\rangle \;=\; \lim_{\varepsilon\to 0} \Im\langle R_0(\lambda^2 \pm i\varepsilon)g, g\rangle \;=\; \pm C\lambda \int_{S^2} |\hat{g}(\lambda\omega)|^2 \sigma(d\omega) \tag{19}$$

where $C \neq 0$ is a constant and $\sigma(d\omega)$ is surface measure on the unit sphere in $\mathbb{R}^3$. It follows that $\hat{g} = 0$ on λS^2, in the sense of L^2 functions.

One of the underlying principles in Agmon [Ag] is that the resolvent $R_0^{\pm}(\lambda^2)$ has special mapping properties when applied to functions whose Fourier transform vanishes on the sphere radius λ. This in turn leads to improved estimates on the decay of $g = -VR_0^{\pm}(\lambda^2)g$. We quote one such statement from the literature:

Lemma 14 ([GS2], section 4). *Let f be a function in $L^1(\mathbb{R}^3)$ such that $\hat{f} = 0$ on the unit sphere. Then*

$$\|R_0^{\pm}(1)f\|_{L^2} \le \frac{1}{\sqrt{8\pi}} \|f\|_{L^1}$$

Scaling considerations dictate that if $\hat{g} = 0$ on λS^2, then $\|R_0^{\pm}(\lambda^2)g\|_{L^2} \le (8\pi\lambda)^{-1/2}\|g\|_{L^1}$.

Returning to the proof of lemma 13, if $\lambda \neq 0$ and $g \in L^{p',1}(\mathbb{R}^3)$ is a solution to $(I + VR_0^{\pm}(\lambda^2))g = 0$, then $f = R_0^{\pm}(\lambda^2)g$ must be an L^2 eigenfunction of $-\Delta + V$. The bootstrapping procedure for g shows that $f \in W^{2,3/2}_{\mathrm{loc}}(\mathbb{R}^3) \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)$. By assumption $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$, so all the hypotheses of Theorem 2 are satisfied. One concludes that $f = 0$, and $g = -Vf = 0$, as desired.

In the case $\lambda = 0$, the expression in (19) is trivially zero, so $\hat{g}$ does not satisfy any additional hypotheses. The resolvent $R_0(0)$ is a bounded map from $L^1(\mathbb{R}^3)$ to $L^{2,-\sigma}(\mathbb{R}^3)$ for any $\sigma > \frac{1}{2}$, however, so $f = R_0(0)g$ is a distributional solution of $-\Delta + V$ which lies in every space $L^{2,-\sigma}(\mathbb{R}^3), \sigma > \frac{1}{2}$. The assumption that zero energy is neither an eigenvalue nor a resonance requires that $f = 0$, thus $g = -Vf = 0$ as well. $\square$

Remark. To be precise, $R_0^{\pm}(\lambda^2)$ maps $L^{1,1}(\mathbb{R}^3)$ to weak-$L^{3,1}(\mathbb{R}^3)$. The calculation in (17), with the appropriate cosmetic changes, is used to bound $\big(\mathrm{meas}\{|R_0^{\pm}(\lambda^2)g(x)| > h/(1+|x|)\}\big)^{1/3}h$ uniformly for all $h > 0$.

The bootstrapping estimates in (18) will be needed in Section 4 in the following form:

Proposition 15. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Then*

$$\begin{aligned}
\|VR_0^{\pm}(\lambda^2)f\|_{L^{2/(2-\varepsilon),1}} &\lesssim \|f\|_{L^{1,1}} \\
\|VR_0^{\pm}(\lambda^2)f\|_{L^{1,1}} &\lesssim \|f\|_{L^{2/(2-\varepsilon),1}} \\
\|(VR_0^{\pm}(\lambda^2))^{k+3}f\|_{L^{1,1}} &\lesssim (1+|\lambda|)^{-k\varepsilon/4}\|f\|_{L^{1,1}}
\end{aligned} \tag{20}$$

Proof. The first two inequalties are precisely what is stated in (18). The last line combines these with (16'). Neither the choice of exponent $p = \frac{2}{2-\varepsilon}$ nor the power of decay in λ are intended to be sharp. For our purposes it will only matter that k can be chosen large enough to make $k\varepsilon/4 > 1$. $\square$

Proposition 16. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. The family of linear maps $VR_0^{\pm}(\lambda^2)$, considered with respect to the operator norm on $L^{p,1}(\mathbb{R}^3)$, $1 \le p < \frac{3}{2}$, depends continuously on the parameter λ. By duality, $R_0^{\mp}(\lambda^2)V$ is a continuous family of maps on $L^{p,-1}$, $p > 3$.*

Proof. Any function $f \in L^{p,1}(\mathbb{R}^3)$ is also integrable, with $\|f\|_{L^1} \lesssim \|f\|_{L^{p,1}}$. It is then possible to differentiate under the integral sign to obtain

$$\Big|\frac{d}{d\lambda} R_0^{\pm}(\lambda^2) f(x)\Big| \;=\; \Big|\int_{\mathbb{R}^3} (\mp 4\pi i)^{-1} e^{\pm i\lambda|x-y|} f(y)\, dy\Big| \;\lesssim\; \|f\|_{L^{p,1}}$$

If V is bounded and has compact support, then $V \in L^{p,1}$ and $\frac{d}{d\lambda}\big[VR_0^{\pm}(\lambda^2)\big]$ will be a bounded operator on $L^{p,1}(\mathbb{R}^3)$ uniformly in λ, which implies continuity.

For general potentials V, approximate V by a bounded, compactly supported potential V' so that $\|V - V'\| < \varepsilon$. Then $\|(V - V')R_0^{\pm}(\lambda^2) f\|_{L^{p,1}} < C\varepsilon\|f\|_{L^{p,1}}$ where $C < \infty$ is the constant in lemma 8. Assume that V is supported in the ball $B(0, R)$ and that $\sup_x |V(x)| = M$. For every $|\nu| \in \mathbb{R}$, the operator $R_0^{\pm}((\lambda+\nu)^2) - R_0^{\pm}(\lambda^2)$ is bounded from $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ with norm $\frac{|\nu|}{4\pi}$, therefore

$$\|V'[R_0^{\pm}((\lambda+\nu)^2) - R_0^{\pm}(\lambda^2)]f\|_{L^{p,1}} \lesssim MR^4|\nu|\|f\|_{L^{p,1}}$$

By the triangle inequality

$$\liminf_{\nu\to 0} \|V[R_0^{\pm}((\lambda+\nu)^2) - R_0^{\pm}(\lambda^2)]f\|_{L^{p,1}} < 2C\varepsilon\|f\|_{L^{p,1}}$$

$\square$

Lemma 17. *Let $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and assume that zero energy is neither an eigenvalue nor a resonance. Then*

$$\sup_{\lambda\in\mathbb{R}} \|(I + VR_0^{\pm}(\lambda^2))^{-1}\|_{L^{p,1}\to L^{p,1}} < \infty \qquad \text{for all} \quad 1 \le p \le 1+\varepsilon \tag{21}$$

Proof. Consider the case $p > 1$. By corollary 10, there exists $\lambda_0 < \infty$ so that the operator norm of $(VR_0^{\pm}(\lambda^2))^2$ will be less than $\frac{1}{2}$ for all $|\lambda| > \lambda_0$. For these large values of λ, the Neumann series

$$(I + VR_0^{\pm}(\lambda^2))^{-1} = \sum_{k=0}^{\infty} (VR_0^{\pm}(\lambda^2))^{2k}(I + VR_0^{\pm}(\lambda^2))$$

converges geometrically and has norm controlled by $(1 + \|VR_0^{\pm}(\lambda^2)\|) \lesssim 1 + \|V\|$. At every point $\lambda \in \mathbb{R}$, lemma 13 and proposition 16 and the continuity of inverses guarantee that $(I + VR_0^{\pm}(\lambda^2))^{-1}$ is norm-continuous in λ. Thus it is bounded on the compact set $[-\lambda_0, \lambda_0]$.

In the case $p = 1$, we claim that $\|(VR_0^{\pm}(\lambda^2))^2\|_{L^{1,1}\to L^{1,1}}$ vanishes as $|\lambda| \to \infty$. A substantially similar result appears in [DanPie], which we reproduce below with the necessary modifications. Assuming this fact, the remaining steps of the above argument follow immediately. $\square$

Proposition 18. *Suppose* $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. *Then*

$$\lim_{\lambda\to\infty} \|(VR_0^{\pm}(\lambda^2))^2\|_{L^{1,1}\to L^{1,1}} = 0 \tag{22}$$

Proof. Suppose V is supported in the ball $B(0,R)$ and satisfies $|V(x)| < M$. Then $\|VR_0^{\pm}(\lambda^2)f\|_{L^{4/3}} \lesssim R^{\frac{5}{4}}\|VR_0^{\pm}(\lambda^2)f\|_{L^3_{\text{weak}}} \lesssim MR^{\frac{5}{4}}\|f\|_{L^{1,1}}$. It follows from proposition 5 that

$$\|(VR_0^{\pm}(\lambda^2))^2 f\|_{L^{1,1}} \lesssim R^{\frac{13}{4}}\|(VR_0^{\pm}(\lambda^2))^2 f\|_{L^4} \lesssim M^2 R^{\frac{9}{2}}\lambda^{-1/2}\|f\|_{L^{1,1}}$$

Stronger decay estimates are possible, but we are not interested here in optimality.

For general potentials V, write $V = V_1 + V_2$ with V_1 bounded and compactly supported and $\|V_2\| < \epsilon$. Then

$$\|(VR_0^{\pm}(\lambda^2))^2\| \le \|(V_1R_0^{\pm}(\lambda^2))^2\| + \|V_1R_0^{\pm}(\lambda^2)V_2R_0^{\pm}(\lambda^2)\| + \|V_2R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)\|$$

All three terms on the right-hand side are smaller than ε when λ is sufficiently large. $\square$

Remark. It is also true that the operators $(I + VR_0^{\pm}(\lambda^2))^{-1}$ are uniformly bounded on the unweighted spaces $L^p(\mathbb{R}^3), 1 \le p < \frac{3}{2}$. The proof follows the same Fredholm-alternative argument, but uses (12) in place of (16) and similar substitutions. The details (with more restrictive hypotheses on V) for $p = 1$ can be found in [DanPie] and for $p = \frac{4}{3}$ in [GS2].

The primary condition on V, that $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, is translation-invariant. Indeed, if $V_y(x) = V(x-y)$ is any translate of V, then $\|V_y\| = \|V\|$. The second condition, that zero energy is neither an eigenvalue nor resonance for $-\Delta + V$, is also preserved under translation. The norm of functions in $L^{p,1}(\mathbb{R}^3)$, however, is clearly affected by translations and cannot even be bounded uniformly. Nevertheless a translation-invariant statement of lemma 17 is still possible.

Lemma 19. *Let* $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ *and assume that zero energy is neither an eigenvalue nor a resonance. Then*

$$\sup_{y\in\mathbb{R}^3}\sup_{\lambda\in\mathbb{R}} \|(I + V_yR_0^{\pm}(\lambda^2))^{-1}\|_{L^{p,1}\to L^{p,1}} < \infty \qquad \text{for all} \quad 1 \le p \le 1+\varepsilon \tag{23}$$

Proof. The mapping $y \mapsto V_y$ is uniformly continuous, that is $\|V_y - V_z\| < \delta$ for every pair of points with $|y - z| > \delta'$. Consequently, the family of operators $V_yR_0^{\pm}(\lambda^2)$ are uniformly continuous with respect to variation in the y parameter. Meanwhile, by proposition 16 this family is also continuous with respect to variation in the λ parameter. It follows that

$$(y,\lambda) \in \mathbb{R}^3 \times \mathbb{R} \mapsto V_yR_0^{\pm}(\lambda^2) \in \mathcal{B}(L^{p,1}(\mathbb{R}^3))$$

is continuous on $\mathbb{R}^3 \times \mathbb{R}$.

The decay estimates (16) and (22) hold uniformly over all translations of V. Thus there exists $\lambda_0 < \infty$ such that $\|(V_yR_0^{\pm}(\lambda^2))^2\| < \frac{1}{2}$ for all $|\lambda| > \lambda_0$ and all $y \in \mathbb{R}^3$. As in the proof of lemma 17, the operator norm of $(I + V_yR_0^{\pm}(\lambda^2))^{-1}$ is controlled uniformly by $1 + \|V\|$ at these points.

Suppose V is supported in the ball $B(0,r)$ and $|y| > 3r$. Given a function $f \in L^{p,1}(\mathbb{R}^3)$, let $f_1 = \chi_{B(y,2r)}f$ and $f_2 = f - f_1$. By construction, $f + V_yR_0^{\pm}(\lambda^2)f = f_2$ outside the ball $B(y,2r)$, thus $\|f + V_yR_0^{\pm}(\lambda^2)f\|_{L^{p,1}} \ge \|f_2\|_{L^{p,1}}$.

Within $B(y,2r)$, we have that $f + V_y R_0^{\pm}(\lambda^2) f = f_1 + V_y R_0^{\pm}(\lambda^2)(f_1 + f_2)$. Thus

$$\|f + V_y R_0^{\pm}(\lambda^2) f\|_{L^{p,1}} \geq \|f_1 + V_y R_0^{\pm}(\lambda^2) f_1\|_{L^{p,1}} - \|V_y R_0^{\pm}(\lambda^2) f_2\|_{L^{p,1}}$$

Since every point $x \in B(y,2r)$ satisfies $r < |x| < 5r$, the weighted and unweighted norms are equivalent. There exists a constant $A > 0$ such that $\|g + VR_0^{\pm}(\lambda^2) g\|_{L^p} \geq A\|g\|_{L^p}$ for all $\lambda \in \mathbb{R}$ and every $g \in L^p(\mathbb{R}^3)$. This is equivalent to the uniform boundedness of $(I + VR_0^{\pm}(\lambda^2))^{-1}$ as operators on $L^p(\mathbb{R}^3)$. By translation invariance, the same estimate holds if V is replaced by any V_y. It follows that

$$\|f_1 + V_y R_0^{\pm}(\lambda^2) f_1\|_{L^{p,1}} \geq \frac{A}{5}\|f_1\|_{L^{p,1}}$$

since the functions on both sides are supported in $B(y,2r)$. For f_2, the crude estimate $|R_0^{\pm}(\lambda^2) f_2(x)| \leq (4\pi r)^{-1}\|f_2\|_{L^1} \lesssim (4\pi r)^{-1}\|f_2\|_{L^{p,1}}$ is valid at all $x \in B(y,2r)$. This suffices to show that

$$\|V_y R_0^{\pm}(\lambda^2) f_2\|_{L^{p,1}} \leq C\|V_y\|_{L^p}\|f_2\|_{L^{p,1}}$$

Applying the triangle inequality to $f = f_1 + f_2$, we conclude that

$$\|f + V_y R_0^{\pm}(\lambda^2) f\|_{L^{p,1}} \geq \max\big(\|f_2\|,\ \frac{A}{5}\|f\| - (\frac{A}{5} + C\|V\|)\|f_2\|\big) \geq \frac{A}{A + 5C\|V\| + 5}\|f\|_{L^{p,1}}$$

Here we are taking advantage of the fact that $\|V\| = \|V_y\|$ Observe that none of the constants in this inequality depend on the size or support of V. Given an arbitrary potential $V \in L^{\frac{3}{2}(1+\varepsilon)} \cap L^1$, it is then possible to choose $V_r = \chi_{|x|<r} V$ so that $\|V - V_r\| \lesssim \frac{A}{2(A+5C\|V\|+5)}$. By lemma 8 and the triangle inequality,

$$\|f + V_y R_0^{\pm}(\lambda^2) f\|_{L^{p,1}} \geq \frac{A}{2(A + 5C\|V\| + 5)}\|f\|_{L^{p,1}}$$

for all $|y| > 3r$.

Having established a uniform bound on $(I + V_y R_0^{\pm}(\lambda^2))^{-1}$ for all $|\lambda| > \lambda_0$ and for all $|y| > 3r$, only a compact region of $\mathbb{R}^3 \times \mathbb{R}$ remains to be considered. However $(I + V_y R_0^{\pm}(\lambda^2))^{-1}$ is a continuous function of (y, λ), hence it is bounded on this domain as well. $\square$

4 Calculations

Our goal at this point is to prove the estimate

$$\begin{aligned}\int_0^\infty e^{it\lambda^2}\lambda\Big\langle\big[R_V^+(\lambda^2)(VR_0^+(\lambda^2))^{m+2} - \ R_V^-(\lambda^2)(VR_0^-(\lambda^2))^{m+2}\big]f, g\Big\rangle\, d\lambda \\ = \int_0^\infty e^{it\lambda^2}\lambda\langle A(\lambda)f, g\rangle\, d\lambda \ \lesssim \ |t|^{-\frac{3}{2}}\|f\|_1\|g\|_1\end{aligned} \tag{24}$$

This will be true if and only if the operator

$$\int_0^\infty e^{it\lambda^2}\lambda\, A(\lambda)\, d\lambda$$

is a well defined map from L^1 to L^∞ whose operator norm is controlled by $|t|^{-3/2}$.

Lemma 20. *For sufficiently large values of m,* $\lim_{\lambda\to\infty} \|A(\lambda)\| = 0$ *as a map from* $L^1(\mathbb{R}^3)$ *to* $L^\infty(\mathbb{R}^3)$.

Proof. Decompose $A(\lambda)$ into a telescoping series

$$\begin{aligned} A(\lambda) \;=\; & \Big[(R_V^+(\lambda^2) - R_V^-(\lambda^2))(VR_0^+(\lambda^2))^{m+2} + \\ & \qquad (R_V^-(\lambda^2)V)\sum_{k=0}^{m+1}(R_0^-(\lambda^2)V)^k(R_0^+(\lambda^2) - R_0^-(\lambda^2))(VR_0^+(\lambda^2))^{m+1-k}\Big] \end{aligned}$$

The difference $R_0^+(\lambda^2) - R_0^-(\lambda^2)$ is precisely a convolution with the kernel $\frac{i\,sin(\lambda|x|)}{2\pi|x|}$. This maps L^1 to L^∞ with operator norm proportional to λ. The difference of perturbed resolvents has a similar bound, by the identity

$$R_V^+(\lambda^2) - R_V^-(\lambda^2) = (I + R_0^+(\lambda^2)V)^{-1}(R_0^+(\lambda^2) - R_0^-(\lambda^2))(I + VR_0^-(\lambda^2))^{-1} \tag{25}$$

Choose m so that $(m-3)\varepsilon > 1$. By (13), each of the $m+2$ terms is a bounded map from L^1 to L^∞ with norm controlled by $\lambda(1+|\lambda|)^{-(m-3)\varepsilon}$. □

We may then integrate by parts to obtain

$$\int_0^\infty e^{it\lambda^2}\lambda\, A(\lambda)\, d\lambda = -\frac{1}{2it}\int_0^\infty e^{it\lambda^2} A'(\lambda)\, d\lambda \tag{26}$$

The boundary term at infinity vanishes by lemma 20. The boundary term at $\lambda = 0$ vanishes because $R_0^+(0) = R_0^-(0)$. From this point forward, cancellation involving $R_0^+(\lambda^2) - R_0^-(\lambda^2)$ will not play a major role; this allows us to express $A'(\lambda)$ in a less cumbersome manner. Recall that

$$A'(\lambda) = \frac{d}{d\lambda}\big[R_V^+(\lambda^2)(VR_0^+(\lambda^2))^{m+2}\big] - \frac{d}{d\lambda}\big[R_V^-(\lambda^2)(VR_0^-(\lambda^2))^{m+2}\big] \;:=\; B^+(\lambda) - B^-(\lambda)$$

For all $\lambda > 0$, the resolvent $R_V^+(\lambda^2)$ may be defined via two different limits, since $R_V(\lambda^2 + i0) = R_V((\lambda + i0)^2)$. If zero energy is neither a resonance nor an eigenvalue, the latter expression admits an analytic continuation into the half-plane $\Im\lambda > 0$, with continuous extension to the boundary satisfying

$$R_V((-\lambda + i0)^2) = R_V((\lambda - i0)^2)$$

The same expression is of course true for the free resolvent as well. A similar analytic extension exists for $B^+(\lambda)$, with the boundary identity $B^+(-\lambda) = -B^-(\lambda)$. The change in sign is a result of differentiation with respect to λ. The right-hand integral in (26) may now be rewritten as

$$-\frac{1}{2it}\int_{-\infty}^{\infty} e^{it\lambda^2} B^+(\lambda)\, d\lambda$$

The proof of theorem 1 concludes with an estimate for this integral.

Lemma 21. *With* $V \in L^{\frac{3}{2}(1+\varepsilon)}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ *and* $B^+(\lambda)$ *defined as above,*

$$\Big\|\int_{-\infty}^{\infty} e^{it\lambda^2} B^+(\lambda) f\Big\|_{L^\infty} \lesssim |t|^{-\frac{1}{2}}\|f\|_{L^1}$$

for all functions $f \in L^1(\mathbb{R}^3)$.

Proof. We may express $B^+(\lambda)$ as an integral operator whose kernel is given formally by the expression

$$B^+(\lambda, x, y) = \frac{d}{d\lambda}\Big\langle (I + R_0^+(\lambda^2)V)^{-1}R_0^+(\lambda^2)(VR_0^+(\lambda^2))^m V(\cdot)\frac{e^{i\lambda|\cdot - x|}}{4\pi|\cdot - x|}, V(\cdot)\frac{e^{-i\lambda|\cdot - y|}}{4\pi|\cdot - y|}\Big\rangle \tag{27}$$

The inner product as written above may not be well-defined because of the local singularites in V. If, however, the derivative is brought inside and applied according to the Leibniz rule, then each term will be finite. Essentially this is because $\frac{d}{d\lambda}R_0^+(\lambda^2)$ is a uniformly bounded operator from $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$, leading to a pairing between the dual spaces $L^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$.

It is sufficient to show that $B^+(\lambda, x, y)$ is uniformly bounded, and $\int_{\mathbb{R}} \big|\frac{d}{d\lambda}\big[e^{-i\lambda|y-x|}B^+(\lambda, x, y)\big]\big|\, d\lambda$ is bounded uniformly in x and y. Then the lemma follows from a stationary-phase argument, estimating the size of the integral

$$\int_{-\infty}^{\infty} e^{it(\lambda + \frac{|y-x|}{2t})^2}\big[e^{-i\lambda|y-x|}B^+(\lambda, x, y)\big]\, d\lambda$$

The derivative in (27) can fall in any of $m + 4$ locations, leading to a sum of four terms:

$16\pi^2 B^+(\lambda, x, y) =$

$$i\Big\langle (I + R_0^+(\lambda^2)V)^{-1}(R_0^+(\lambda^2)V)^{m+1}e^{i\lambda|\cdot - x|},\ \frac{V(\cdot)e^{-i\lambda|\cdot - y|}}{|\cdot - y|}\Big\rangle \tag{27a}$$

(27b)

$$+\sum_{k=0}^{m}\Big\langle (I + R_0^+(\lambda^2)V)^{-1}(R_0^+(\lambda^2)V)^k\Big[\frac{d}{d\lambda}R_0^+(\lambda^2)\Big](VR_0^+(\lambda^2))^{m-k}\frac{V(\cdot)e^{i\lambda|\cdot - x|}}{|\cdot - x|},\ \frac{V(\cdot)e^{-i\lambda|\cdot - y|}}{|\cdot - y|}\Big\rangle$$

(27c)

$$-\Big\langle (I + R_0^+(\lambda^2)V)^{-1}\Big[\frac{d}{d\lambda}R_0^+(\lambda^2)\Big](I + VR_0^+(\lambda^2))^{-1}(VR_0^+(\lambda^2))^{m+1}\frac{V(\cdot)e^{i\lambda|\cdot - x|}}{|\cdot - x|},\ \frac{V(\cdot)e^{-i\lambda|\cdot - y|}}{|\cdot - y|}\Big\rangle$$

(27d)

$$+\,i\Big\langle (I + VR_0^+(\lambda^2))^{-1}(VR_0^+(\lambda^2))^{m+1}\frac{V(\cdot)e^{i\lambda|\cdot - x|}}{|\cdot - x|},\ e^{-i\lambda|\cdot - y|}\Big\rangle$$

The formula in (27c) is a consequence of the chain rule $\frac{d}{d\lambda}M^{-1}(\lambda) = M^{-1}(\lambda)\big[\frac{d}{d\lambda}M(\lambda)\big]M^{-1}(\lambda)$ for operator-valued functions, and also the commutator relation $V(I+R_0^+(\lambda^2)V)^{-1} = (I+VR_0^+(\lambda)^2)^{-1}V$.

Each of the four terms is bounded uniformly in (λ, x, y) by some combination of (12), (12'), Hölder's inequality, and the following observations:

$\sup_x \|e^{i\lambda|\cdot - x|}\|_\infty = 1$.

$\sup_x \|V(\cdot)|\cdot - x|^{-1}\|_1 \lesssim \|V\|$.

$\sup_\lambda \|(I + VR_0^+(\lambda^2))^{-1}\|_{1\to 1} < \infty$. The proof is essentially identical to that of lemma 17.

$\sup_\lambda \big\|\frac{d}{d\lambda}R_0^+(\lambda^2)\big\|_{1\to\infty} = (4\pi)^{-1}$.

We now turn our attention to the second assertion, that $\sup_{(x,y)}\int_{\mathbb{R}}\big|\frac{d}{d\lambda}\big[e^{-i\lambda|y-x|}B^+(\lambda, x, y)\big]\big|\, d\lambda < \infty$. In fact we will prove the pointwise estimate

$$\Big|\frac{d}{d\lambda}\big[e^{-i\lambda|y-x|}B^+(\lambda, x, y)\big]\Big| \lesssim (1 + |\lambda|)^{-(m-6)\varepsilon/4} \quad \textit{for all } m \geq 8. \tag{28}$$

so that it suffices to choose $m > \frac{4}{\varepsilon} + 6$. For the sake of brevity, we will only calculate explicitly the derivatives associated to a typical term in the expression (27b). The same techniques apply equally well to each of the other terms.

Suppose the derivative falls anywhere except on the already-differentiated resolvent, a typical example being

$$\Big\langle (I + R_0^+(\lambda^2)V)^{-1}(R_0^+(\lambda^2)V)^\ell \Big[\frac{d}{d\lambda}R_0^+(\lambda^2)\Big] V(R_0^+(\lambda^2)V)^{k-\ell-1} \\ \Big[e^{-i\lambda|x-y|}\frac{d}{d\lambda}R_0^+(\lambda^2)\Big](VR_0^+(\lambda^2))^{m-k}\frac{V(\cdot)e^{i\lambda|\cdot-x|}}{|\cdot-x|},\ \frac{V(\cdot)e^{-i\lambda|\cdot-y|}}{|\cdot-y|}\Big\rangle$$

Using (13) and the four observations listed above, this term is seen to be less than $(1+|\lambda|)^{-(m-7)\varepsilon}$. Of particular note here is the fact that multiplication by V is a bounded map between $L^\infty(\mathbb{R}^3)$ and $L^1(\mathbb{R}^3)$.

The case where the derivative falls on $(I + R_0^+(\lambda^2)V)^{-1}$ has only superficial differences, since the operator

$$\frac{d}{d\lambda}(I + R_0^+(\lambda^2)V)^{-1} = (I + R_0^+(\lambda^2)V)^{-1}\Big[\frac{d}{d\lambda}R_0^+(\lambda^2)\Big](I + VR_0^+(\lambda^2))^{-1}V$$

is still bounded on $L^\infty(\mathbb{R}^3)$ uniformly in λ.

In order to address the case where both derivatives fall in the same place, we use estimates in $L_x^{p,\sigma}(\mathbb{R}^3)$, the weighted norm space defined by

$$\|f\|_{L_x^{p,\sigma}} := \|(1+|\cdot-x|)^\sigma f\|_{L^p}$$

This cannot easily be avoided, as the kernel of $\dfrac{d^2}{d\lambda^2}R_0^+(\lambda^2)$ experiences polynomial growth in the spatial variables. Note that $L_x^{p',-\sigma}(\mathbb{R}^3)$ is the dual space to $L_x^{p,\sigma}(\mathbb{R}^3)$ for any $1 \le p < \infty, \sigma \in \mathbb{R}$.

It is clear by translation that the action of $VR_0^+(\lambda^2)$ on $L_x^{p,1}$ is equivalent to that of $V_{-x}R_0^+(\lambda^2)$ acting on $L^{p,1}$. The bounds in lemma 8 and its corollaries (in particular, (20)) therefore hold on all spaces $L_x^{p,1}(\mathbb{R}^3)$ with p in the appropriate range. Similarly, lemma 19 asserts that $(I + VR_0^\pm(\lambda^2))^{-1}$ is bounded on all $L_x^{p,1}(\mathbb{R}^3)$, uniformly in x.

Two other observations are worth noting at this point. First is the norm bound

$$\Big\|\frac{V(\cdot)}{|\cdot-x|}\Big\|_{L_x^{1,1}} \lesssim \|V\|$$

which holds for all $x \in \mathbb{R}^3$. Second, the operator $\frac{d}{d\lambda}\big[e^{-i\lambda|y-x|}\frac{d}{d\lambda}R_0^+(\lambda^2)\big]$ maps $L_x^{1,1}$ to $L_y^{\infty,-1}$. This is seen by examining the integration kernel

$$|K(x_2,x_1)| = \Big|\frac{d}{d\lambda}e^{i\lambda(|x_2-x_1|-|y-x|)}\Big| \le |x_2-y| + |x_1-x|$$

Clearly $\sup\limits_{x_1,x_2}\big|(1+|x_2-y|)^{-1}K(x_2,x_1)(1+|x_1-x|)^{-1}\big| \le 2$.

We now return to the remaining term in $\frac{d}{d\lambda}\big[e^{-i\lambda|x-y|}B^+(\lambda,x,y)\big]$, namely:

$$\Big\langle (I + R_0^+(\lambda^2)V)^{-1}(R_0^+(\lambda^2)V)^k\Big[\frac{d}{d\lambda}\big(e^{-i\lambda|y-x|}\frac{d}{d\lambda}R_0^+(\lambda^2)\big)\Big](VR_0^+(\lambda^2))^{m-k}\frac{V(\cdot)e^{i\lambda|\cdot-x|}}{|\cdot-x|},\ \frac{V(\cdot)e^{-i\lambda|\cdot-y|}}{|\cdot-y|}\Big\rangle$$

By (20), its dual, and the above mapping estimate for the twice-differentiated resolvent, the left-hand function is in $L^{\infty,-1}_y(\mathbb{R}^3)$ with norm less than $(1+|\lambda|)^{-(m-6)\varepsilon/4}$. The right-hand function is in the dual space $L^{1,1}_y(\mathbb{R}^3)$ with norm controlled by $\|V\|$.

The pairing of these two functions is therefore finite, and controlled pointwise in λ by the integrable expression $(1+|\lambda|)^{-(m-6)\varepsilon/4}$. □

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